Average Volumetric Concentration in Two-Phase Flow Systems

A general expression which can be used either for predicting the average volumetric concentration or for analyzing and interpreting experimental data is derived. The analysis takes into account both the effect of nonuniform flow and concentration profiles as well as the effect of the local relative velocity between the phases. The first effect is taken into account by a distribution parameter, whereas the latter is accounted for by the weighted average drift velocity. Both effects are analyzed and evaluated. The results predicted by the analysis are compared with experimental data obtained for various two-phase flow regimes, with various liquid-gas mixtures in adiabatic, vertical flow over a wide pressure range. Good agreement with experimental data is shown.

Introduction

Present Status

The ability to predict the volumetric concentration of a phase, i.e., the holdup or the void fraction as function of the design and operating parameters (geometry, pressure, flow rates, thermodynamic and transport properties of the phases, etc.), is of considerable importance to the nuclear reactor technology and to the chemical process industry. Consequently, numerous publications, dealing with both the experimental and the theoretical aspects of the problem, have appeared in the literature. However, with the exception of a few papers discussed herein, this substantial effort has not contributed significantly either toward the understanding of the physical processes involved or toward helping the designer in providing him with design information and criteria of sufficient accuracy, reliability, and generality. This state of the art and knowledge is the result of the experimental and analytical approaches which have been heretofore used in analyzing the problem.

With the exception of some of the experiments to be discussed, few experiments have been either properly designed to provide basic information (although many investigations have claimed this purpose) or properly instrumented to provide the kind of data that are required for the solution of the problem. Most experiments were conducted with the purpose of obtaining data for a particular design, paying but little attention to other measurement and information which were outside of the designer’s immediate need. Such data, obtained from a large number of experiments, were often correlated by means of computers, thus providing correlations void of any physical significance.

With exception of some of the papers discussed here, most of the published analyses are based on models and formulations which have been obtained by means of high-speed computers. It is apparent from this brief discussion that a general method for predicting the volumetric concentration or for interpreting experimental results is not yet available.

Purpose of the Paper

It is the purpose of this paper to provide a general method which can be used either for predicting the volumetric concentration or for analyzing and interpreting experimental data.

The analysis takes into account the effect of the nonuniform flow and concentration distributions across the duct as well as the effect of the local relative velocity between the two phases. The results are general and can be applied to any two-phase flow regime. In this paper, we shall apply them to adiabatic, dispersed two-phase flow systems with fully established, i.e., constant, velocity and concentration profiles. Two-phase flow systems with heat and/or mass addition or removal along the duct can be added to a two-phase mixture, for example, by blowing air through a porous wall of the duct.

Nomenclature

Units in the \([M, L, T]\) System

- \(A\) = flow area \([L^2]\)
- \(C_d\) = distribution parameter \([0]\)
- \(C_f\) = terminal velocity coefficient (see equation (10)) \([0]\)
- \(D\) = pipe dia \([L]\)
- \(d\) = bubble dia \([L]\)
- \(g\) = gravitational acceleration \([L/T^2]\)
- \(I_0\) = integral as defined by equation (63)
- \(k\) = exponent used in drift velocity (see equation (57)) \([0]\)
- \(K\) = \((\alpha)/\beta\) = flow parameter \([0]\)
- \(m\) = exponent on velocity distribution (see equation (43)) \([0]\)
- \(n\) = exponent on void distribution (see equation (44)) \([0]\)
- \(P\) = pressure \([M/LT^2]\)
- \(Q\) = volumetric flow rate \([L^2/T]\)

- \(R\) = pipe radius \([L]\)
- \(r\) = radial variable \([L]\)
- \(r^*\) = \(r/R\) dimensionless radial variable \([0]\)
- \(t\) = time \([T]\)
- \(v_f\) = \(v_i - \langle j\rangle\) = drift velocity \([L/T]\)
- \(v\) = velocity \([L/T]\)
- \(v_r\) = \(v_i - v_f\) = relative velocity \([L/T]\)
- \(j_i\) = volume flux density \([L^3/T^2]\)
- \(\langle j\rangle\) = \((j_i + j_v)/A\) = average volumetric flux density of the mixture \([L^3/T^2]\)

(Continued on next page)
will be considered in other publications, because in these systems neither the velocity profile nor the concentration profile remains constant but both change as the mixture flows through the duct.

The results presented in this paper show how the flow distribution, the thermodynamic and transport properties of the two phases, the system pressure, the duct geometry, and the operating conditions affect the average value of the volumetric concentration. Therefore, in addition to presenting a general expression for the volumetric concentration, the analysis is useful because it demonstrates the kind of experimental data and the kind of measurements which are required if an accurate solution of the problem is desired.

Previous Work

There are two effects which must be taken into account in an analysis of the volumetric concentration problem. One must consider the effect of the local relative velocity between the phases as well as the effect of the nonuniform flow and concentration distribution across the duct. In the literatures, these two effects have been considered separately.

Behringer [3], in 1936, was apparently the first to consider the effect of the local relative velocity between the phases. Neglecting the effect of nonuniform flow and concentration distribution across the duct, he derived from continuity considerations the following expression for the velocity $v_2$ of a bubble in a bubbly mixture:

$$v_2 = \frac{Q_1}{A} + \frac{Q_2}{A} + v_o$$  \hspace{1cm} (1)

where $Q_1$ and $Q_2$ are the volumetric flow rates of the liquid and the gas, and $v_o$ is the terminal rise velocity of a single bubble in an infinite medium. The volumetric concentration $\alpha$ then follows from the relation between $v_2$ and the superficial velocity of the gas; thus:

$$\alpha = \frac{Q_i}{v_1}$$ \hspace{1cm} (2)

In his 1936 paper, Behringer reported good agreement of values predicted by equations (1) and (2) with his experimental data.

Bankoff [4] was apparently the first to consider the effect of the radial nonuniform flow and volumetric concentration in the bubbly two-phase flow regime. Neglecting the effect of the local relative velocity between the phases but accounting for the nonuniform profiles, he obtained the following relation between the mean velocities of the two phases and the volumetric concentration:

$$\frac{v_2}{v_1} = \frac{\langle j_2 \rangle / \langle \alpha \rangle}{\langle j_1 \rangle / (1 - \langle \alpha \rangle)} = 1 - \langle \alpha \rangle$$ \hspace{1cm} (3)

where $K$ is a flow parameter which is determined by the expression

$$K = \frac{\langle \alpha F \rangle}{\langle \alpha \rangle} = \frac{\langle \alpha F \rangle}{\langle \alpha \rangle}$$ \hspace{1cm} (4)

It is rather surprising, in view of the numerous incorrect formulations which have subsequently appeared in the literature, that the formulation of Behringer was neither used nor referred to. This is even more surprising since an AEC translation of his paper is available.

Components of the velocity and concentration profiles, whereas $\langle j_1 \rangle$ and $\langle j_2 \rangle$ are the "superficial" velocities of the two phases. The flow parameter $K$ is a function of pressure, quality, and mass flow rate. Because of insufficient experimental information, the value of $K$ was not directly predicted by Bankoff. However, he was able to show analytically that, for circular pipes, the effective range of variation of $K$ is from about $K = 0.5$ to $K = 1.0$. For an assumed value of $K = 0.80$, good agreement was reported with Martinelli-Nelson correlation over a pressure range from 100–2500 psia.

It was shown by Zaher [5] that the flow parameter $K$ in Bankoff's analysis is identical to the ratio of the volumetric concentration $\langle \alpha \rangle$ to the "flowing volumetric concentration" $\langle \beta \rangle$; thus:

$$\frac{\langle \alpha \rangle}{\langle \beta \rangle} = K$$ \hspace{1cm} (5)

where the "flowing volumetric concentration" $\langle \beta \rangle$ is defined by

$$\langle \beta \rangle = \frac{Q_1/A}{Q_1/A + Q_2/A}$$ \hspace{1cm} (6)

Equation (4), obtained first empirically by Arinardi [6], was used extensively in the Russian literature to correlate experimental data obtained at high mass flow rates over a large pressure range. The dependence of $K$ upon pressure as determined from these experiments is given in [4, 5, and 6]. Since Bankoff's analysis neglects the effect of the local relative velocity, it can be expected that equation (4) will be successful in correlating the data only if this effect is negligible.

Following the work of [3 and 4], there have been numerous papers which took into consideration the effect of relative velocity and/or have attempted to take into account the effect of the nonuniform flow distribution as well.

An important contribution was made by Griffith and Wallis [7], who, for the slug flow regime, rederived equation (1) using again continuity considerations and expressing the terminal bubble rise velocity $v_o$ by the velocity of the Dimitrescu-Taylor bubble, i.e., by

$$v_o = 0.35 \left[ \frac{g \Delta \rho D}{\mu} \right]^{1/2}$$ \hspace{1cm} (7)

This analysis has been since extended by Moisio and Griffith [8] to consider the entrance and flow transition effects and by Griffith [9] to take into consideration the effects of heat addition.

Two attempts to account for the effect of nonuniform flow distribution on the volumetric concentration in the slug flow regime were reported by Nicklin, Wilkes, and Davidson [10] and by Neal [11]. Both papers modify equation (1) by introducing a constant $C_0$ to account for the nonuniform distribution. Thus both papers present an equation of the form of

$$v_2 = C_0 \frac{Q_1 + Q_2}{A} + 0.35 \left[ \frac{g \Delta \rho D}{\mu} \right]^{1/2} \hspace{1cm} (8)$$

Whereas Nicklin, Wilkes, and Davidson consider the velocity $v_2$ to be the actual velocity of the vapor slug, Neal considers it to be "the total cross-sectional average gas velocity."

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**Nomenclature**

$N_u =$ Weber number [0]  
$\alpha =$ volume concentration [0]  
$\beta =$ flowing volume concentration [0]  
$\rho =$ density $[ML^{-3}]$  
$\Delta \rho =$ $\rho_i - \rho_l [ML^{-3}]$  
$\sigma =$ surface tension $\left[ ML^{-1} T^{-1} \right] = \left[ M T^{-2} \right]$  
$\mu =$ dynamic viscosity $[ML^{-1}T^{-1}]$  
$F =$ point quantity  
$\langle F \rangle =$ weighted mean value

**Subscripts**

0 = orifice  
1 = liquid phase  
2 = gas phase  
$m =$ mixture  
$\infty =$ terminal in an infinite medium  
$r =$ relative  
$i =$ terminal  
$j =$ number of the phase  
$A =$ with respect to the center of volume
The authors of [10] use plausible arguments to select the value of the constant \( C_0 \) to be

\[ C_0 = 1.2 \]  

instead of unity as given by equation (1). They justify this modification by arguing that the bubble is located in the high-velocity region and is transported therefore faster than the average flow. The value of \( C_0 = 1.2 \) comes from the fact that the ratio of the maximum to the average flow velocity in turbulent flow is equal to approximately 1.2. We note that the effect of the concentration profile was not accounted for by the authors of [10] in computing the value of \( C_0 \).

Neal [11], on the other hand, notes that the value of \( C_0 \) can be obtained from Bankoff’s analysis and expressed it therefore by the inverse of the flow parameter \( K^* \); thus

\[ C_0 = \frac{1}{\int j \, da} \]  

where \( j \) is the local volumetric flux density of the mixture. He notes also that, in addition to flow distribution effects, one must consider the effect of local relative velocity. In order to account for this effect, he adds, without proof, the second term on the right-hand side of equation (7).

Finally, Street and Tek [12] present an equation for the slug flow regime of the form of

\[ v_s = C_0 \frac{Q_1 + Q_2}{A} + C_1 [pD]^{1/2} \]  

where now both constant \( C_0 \) and \( C_1 \) depend on the flow distribution.

In this paper, we shall derive a general expression for the volumetric concentration applicable to any two-phase flow regime. From this expression, one can obtain equation (7) as a special case of the slug flow regime. Other expressions applicable to other flow regimes are also presented and discussed.

Analysis

Velocity Fields and Continuity Considerations

A more detailed discussion of the various velocity fields which are useful for characterizing the flow of a two-phase mixture is given in [13]. Here we shall use some of these expressions to formulate and solve our problem. We consider here a three-dimensional problem and express the velocities in terms of vectors.

In analogy with the kinetic theory of gases or with plasmas, we define the local number velocities or, more appropriately to this problem, the volumetric flux densities by

\[ j_1 = \alpha v_2 \]  
\[ j_1 = (1 - \alpha) v_1 \]  

the relative velocity between the two phases by

\[ v_{ij} = v_i - v_j \]  

and the diffusion or drift velocities with respect the volumetric flux density of the mixture by

\[ V_{ij} = \bar{v}_{ij} - j \]  
\[ V_{ij} = v_i - j \]  

where the volumetric flux density of the mixture is defined by

\[ j = j_1 + j_2 \]  

In this paper, we consider a two-phase flow system in which a change of phase does not take place (either due to evaporation, condensation, flashing, or chemical reaction). For this case, the continuity equations for the two phases are given by

\[ \frac{\partial (1 - \alpha) p_1}{\partial t} + \text{div} \left[ p_1 (1 - \alpha) v_1 \right] = 0 \]  
\[ \frac{\partial (\alpha p_2)}{\partial t} + \text{div} \left[ p_2 \alpha v_2 \right] = 0 \]  

Assuming further constant densities and using equation (11) and (12), we obtain from equations (17) and (18)

\[ \frac{\partial (1 - \alpha) v_1}{\partial t} + \text{div} \, j_1 = 0 \]  
\[ \frac{\partial \alpha v_2}{\partial t} + \text{div} \, j_2 = 0 \]  

when, upon eliminating the volumetric concentration, we obtain

\[ \text{div} \, (j_1 + j_2) = 0 \]  

or, in view of equation (16), we have

\[ \text{div} \, j = 0 \]

It follows therefore that the volumetric flux densities of the two phases do not depend upon space coordinates but may depend only upon time; thus

\[ j_1 + j_2 = j(t) \]

for one-dimensional flow or for multidimensional irrotational flows.

We shall now show that, if the local relative velocity \( v_r \) is zero, then both drift velocities \( V_{ij} \) and \( V_{ji} \) are zero and the two phases have the same velocity which is equal to the volumetric flux density of the mixture.

By means of equations (11) through (16), we can express the drift velocities of the two phases in terms of the relative velocity \( v_{ij} \); thus

\[ V_{ij} = \alpha v_r (1 - \alpha) \]  
\[ V_{ij} = -v_r \alpha \]  

It can be seen from equations (24) and (14) that, when the relative velocity is zero, i.e., when

\[ v_r = 0 \]  

then

\[ V_{ij} = 0 \]  
\[ v_i = j \]

Similarly, when equation (26) holds, then it follows from equations (15) and (25) that

\[ V_{ij} = 0 \]  
\[ v_i = j \]

Average Velocity and Weighted Mean Velocity of the Gas

In two-phase flow systems, one has more often data on average values than on the local ones. Consequently, it is advantageous to consider the average value of a scalar or of a vector quantity \( F \) over the cross-sectional area of the duct defined by

\[ \langle F \rangle = \frac{1}{A} \int_A F \, dA \]

Introducing the expressions for the local values of the local velocities \( v \) given by equation (14) into equation (31), we obtain the average velocities (averaged over the cross-sectional area of the duct) of the gas; thus

\[ \langle v \rangle = \langle \frac{j}{\alpha} \rangle = \langle j \rangle + \langle V_{ij} \rangle \]

where we have taken also into account equation (11).

Although equation (32) may be useful in some analyses of two-phase flow systems, it is more advantageous to formulate the problem by considering volumetric flux density \( j_\alpha \) instead of the velocity \( v \). The reason for this becomes obvious if one considers that the system input parameters readily available to a
designer or to an experimenter, are the average volumetric flux densities defined by

$$\langle j \rangle = \langle \alpha \omega \rangle = \frac{Q}{A} \quad (33)$$

This average velocity is often referred to as the "superficial" velocity.⁶

In view of these relations, we are led to consider the weighted mean value of the quantity \( F \), defined by

$$F = \frac{1}{A} \int_A \alpha F \, dA$$

Whence we obtain the weighted mean velocity \( \overline{v}_2 \) of the gas phase, thus

$$\overline{v}_2 = \frac{\langle \alpha \omega \rangle}{\langle \alpha \rangle} = \frac{\langle j \rangle}{\langle \alpha \rangle} \quad (35)$$

Note that this definition is identical to the definition of the mean velocity of particles given in the kinetic theory of gases and liquids. In view of equation (14), the weighted mean velocity also can be expressed as

$$\overline{v}_2 = \frac{\langle \omega \rangle}{\langle \alpha \rangle} + \left( \frac{\alpha V_{S_2}}{\langle \alpha \rangle} \right) \quad (36)$$

It cannot be overemphasized that, in general, the average velocity \( \overline{v}_2 \) defined by equation (32) is not equal to \( \overline{v}_2 \) defined by equation (35), i.e., by equation (36).

The General Expression for the Average Volumetric Concentration

The weighted mean velocity \( \overline{v}_2 \) given by equation (36), can be cast in several forms which are most useful for analyzing experimental data and for determining the average volumetric concentration \( \langle \alpha \rangle \). Thus, multiplying and dividing the first term on the right-hand side of equation (36) by \( \langle j \rangle \), we obtain

$$\overline{v}_2 = \frac{\langle j \rangle}{\langle \alpha \rangle} = C_0 \langle j \rangle + \frac{\alpha V_{S_2}}{\langle \alpha \rangle} \quad (37)$$

where the distribution parameter \( C_0 \) is defined by

$$C_0 = \frac{\langle \omega \rangle}{\langle \alpha \rangle} = \frac{1}{A} \int_A \alpha \omega \, dA$$

(38)

The inverse of this parameter first appeared in Bankoff's analysis [4], where it was called the flow parameter \( K \) (see section "Previous Work").

Equation (38) can be expressed in a nondimensional form by dividing both sides by \( \langle j \rangle \), thus

$$\frac{\langle \beta \rangle}{\langle \alpha \rangle} = C_0 + \frac{\langle \alpha V_{S_2} \rangle}{\langle \alpha \rangle} \quad (39)$$

where the average volumetric flow concentration \( \langle \beta \rangle \) (or "flowing concentration") is defined by

$$\langle \beta \rangle = \frac{\langle j \rangle}{\langle \alpha \rangle} = \frac{Q_2}{Q_1 + Q_2} \quad (40)$$

This quantity, which is known a priori, depends only upon the operating conditions.

⁶ We prefer to use the term average volumetric flux density instead of "superficial" velocity because (a) it brings up the true significance and physical meaning of \( j \), and (b) it relates it to expressions well known in thermodynamics and rational mechanics. The expression "superficial" velocity has no meaning.

The general expression for predicting the average volumetric concentration \( \langle \alpha \rangle \) follows then from equation (30); thus

$$\langle \alpha \rangle = \frac{\langle \beta \rangle}{C_0 + \langle \alpha V_{S_2} \rangle / \langle \alpha \rangle} \quad (41)$$

Using equation (41) and the definitions of the weighted mean velocities, we can form the velocity ratio; thus

$$\frac{\overline{v}_2}{\langle \alpha \rangle} \langle \alpha \rangle = \frac{\langle j \rangle}{\langle \alpha \rangle} \langle \alpha \rangle = \frac{1 - \alpha}{C_0 + \langle \alpha V_{S_2} \rangle / \langle \alpha \rangle} \quad (42)$$

In contrast to previous analyses (often incorrect) and semiempirical results, we have derived here an expression (equations (41), (39), or (37)) which is applicable to any two-phase flow regime. Furthermore, the analysis takes into account both the effect of nonuniform flow and concentration profiles and the effect of the local relative velocity. The first effect is accounted for by the distribution parameter \( C_0 \), whereas the second one is accounted for by the weighted mean drift velocity \( \langle \alpha V_{S_2} \rangle / \langle \alpha \rangle \). For each particular flow regime, the value of the average volumetric concentration \( \langle \alpha \rangle \) can be obtained from equation (41) by inserting the appropriate velocity and concentration profiles and the appropriate expression for the drift velocity. In what follows, we shall apply equations (37) or (41) to various flow regimes. However, before doing this, we consider first, in more detail, the effect of the nonuniform velocity and concentration profiles and the effect of the concentration distribution on the drift velocity.

The Effect of Nonuniform Flow and Concentration Distribution

We shall investigate now the effects of nonuniform flow and concentration distribution on the value of the coefficient \( C_0 \) which we shall henceforth refer to as the distribution parameter. For simplicity, we consider an axially symmetric flow through a circular duct and assume that the flow and concentration distributions are given by

$$\frac{j}{j_c} = 1 - \left( \frac{r}{R} \right)^n \quad (43)$$

and

$$\frac{\alpha - \alpha_c}{\alpha_0 - \alpha_c} = 1 - \left( \frac{r}{R} \right)^n \quad (44)$$

where the subscripts \( c \) and \( w \) refer to the values evaluated at the center line and at the wall of the circular duct.⁷ In the last section of this paper, we discuss the method for determining the values of the exponents \( m \) and \( n \); here we insert equations (43) and (44) into equation (38) and obtain the following expressions for the distribution parameter \( C_0 \)

$$C_0 = 1 + \frac{2}{m + n + 2} \left[ 1 - \frac{\alpha_c}{\langle \alpha \rangle} \right] \quad (45)$$

when expressed in terms of the volumetric concentration \( \alpha_c \) at the wall or when

$$C_0 = \frac{m + 2}{m + n + 2} \left[ 1 + \frac{\alpha_c}{\langle \alpha \rangle} \right] \quad (46)$$

expressed in terms of the volumetric concentration \( \alpha_c \) at the center line.⁸

⁷ We could have assumed other, more complicated, profiles. However, for the purpose of this paper, those considered here are sufficiently illustrative of the physical processes which take place, most often, in vertical flow.

⁸ The relationship between \( \alpha_0 \) and \( \alpha_c \) is given by equation (62) and, therefore, \( C_0 \) can be written in terms of \( \alpha_0 \).
First, we note that, if the concentration is uniform across the duct, i.e., if
\[ a_w = a_c = \langle a \rangle \]  
then it follows from equations (44) and (45) that
\[ C_0 = 1 \]  
If the concentration at the center line is larger than that at the wall, i.e., if
\[ a_c > a_w \]  
then
\[ C_0 > 1 \]  
Finally, if the concentration at the center line is smaller than that close to the wall, i.e., if
\[ a_c < a_w \]  
then
\[ C_0 < 1 \]  
In order to examine the sensitivity of the distribution parameter \( C_0 \), we follow Bankoff and Neal and assume various laminar and turbulent profiles, which are illustrated in Fig. 1 together with the value of \( C_0 \), computed from equation (45). It can be seen that, as soon as \( a_c \ll a_w \), then \( C_0 \) attains a constant value which depends only on the type of the flow and of the concentration profiles. For pronounced parabolic profiles (akin to laminar profiles) (curve I in Fig. 1), the distribution parameter attains a value of \( C_0 = 1.5 \); whereas for flat profiles, it tends to reach a value of unity. Since Bankoff and Neal considered only the case \( a_w = 0 \), our results for the region \( a_w < a_c \) are similar to theirs.

We conclude from the preceding that:
1. The value of the distribution parameter \( C_0 \) depends on the flow and concentration profiles.
2. For fully established profiles, in axisymmetric two-phase flow, this value may range from about \( C = 1.5 \) to \( C = 1.0 \) when \( a_w < a_c \).
3. For fully established profiles when \( a_w > a_c \), the distribution parameter has a value smaller than unity, i.e., \( C_0 < 1 \).

**Effect of Local Relative Velocity**

In order to analyze the effect of the local relative velocity \( v \), between the phases, it is advantageous to formulate the problem by considering the drift velocity \( V_d \) instead of the relative velocity \( v \). Indeed, the **cardinal question** in two-phase flow is concerned with determining the correct form of the drift velocity \( V_d \). In order to answer this question, it is necessary to specify the mode of momentum transfer between the phases. This transfer depends upon the stress fields in each phase as well as upon the geometry of the interface between the two phases. Thus, the question pertaining to the correct expression for the drift velocity is, in essence, the question pertaining to the correct constitutive equation for the mixture.

By examining the form of equation (41), it becomes apparent now why a change of the two-phase flow regime (a change in the geometry of the interface) will affect the value of the average volumetric concentration. It becomes also evident why analyses which do not pay any attention to the flow regime cannot be successful in predicting accurately the value of \( \langle a \rangle \).

In this paper, we consider dispersed two-phase flow regimes with either spherical interfaces between the continuous and dispersed phase or an interface in the form of bullet-shaped slugs, i.e., the slug flow regime. We analyze the general problem by permitting the drift velocity to be a function of concentration. The separated flow regimes—like the annular and the annular-mist flow—will be considered in more detail in other publications. In the section on "Two-Component Systems" of this paper, we show, briefly, that the analysis is applicable to these two flow regimes as well.

For the systems just described, a method for determining the drift velocity is given by Zuber [14]. Briefly, the two-phase mixture is considered as a continuum whose thermodynamic and transport properties depend upon the thermodynamic and transport properties of each phase as well as upon the concentration. The problem is formulated in terms of the momentum equation for the two-phase mixture, two continuity equations, and the equation of motion of a particle. As discussed in [14], the formu-
tion takes into account the effects of both the motion and the presence of other particles on the drag force acting on the representative one. The method has been applied to gas-liquid systems [15], solid-fluid [14, 16], and fluid-fluid systems [17], where it is shown that the predicted results are in satisfactory agreement with experimental data and the semiempirical formulations of [15, 19].

From the definition of the local drift velocity (given by equation (14)) we see that it represents the local velocity of the particle with respect to the local volumetric flux density of the mixture. The simplest expression we can obtain for this velocity is to assume that it is unaffected by concentration, i.e., by the presence of other particles. In such a case, the drift velocity is equal to the terminal velocity of the particle rising in an infinite medium. Indeed, this is true for the slug flow regime, as was shown by the results of [7, 8, 9, 10, 11, 12], and for the bubbly flow in a turbulent stream, as it can be concluded from the experimental results of [15 and 20]. For these conditions, we obtain for the local drift velocity of the gas the following expressions:

\[ V_{lg} = v_l - j = 0.35 \left[ \frac{g \Delta \rho D}{\rho_l} \right]^{1/4} \] (53)

for the slug flow regime

\[ V_{lg} = v_l - j = 1.53 \left[ \frac{g \Delta \rho}{\rho_l^2} \right]^{1/4} \] (54)

for the churn-turbulent bubbly flow.

We note that equation (53) is valid for the slug flow regime when the viscous effects can be neglected; this is certainly the case for water in most systems of practical interest. The range of validity of equation (53) is discussed by White and Beadmore [21]. They present also expressions for the terminal velocity of a slug which should be used in place of equation (53) when the viscous effects become important.

We note further that the value of the constant equal to 1.53 in equation (54) is that proposed by Harmathy [22], whereas Peebles and Garber [23] recommend a value equal to 1.18. Both values, as well as equation (54), are approximations. The usefulness of equation (54) lies in the fact that the rise velocity is independent of the bubble diameter, which is not known a priori.

When the presence of other particles (bubbles) affects the motion of a given bubble, then the local drift velocity will depend on the concentration. This problem has been already analyzed by Zuber and Hench [15], who present the following expressions for the drift velocity:

\[ V_{lg} = \frac{g \Delta \rho D^3}{18 \mu_l (1 - \alpha)^k} \] (55)

for small bubbles obeying the Stokes law, i.e., for bubbles with a diameter less than \(0.5 \times 10^{-4}\) cm; whereas for larger bubbles with diameters of the order of \(10^{-1}\) to 2 cm, the local drift velocity is given by

\[ V_{lg} = 1.53 \left[ \frac{g \Delta \rho}{\rho_l^2} \right]^{1/4} \frac{1}{(1 - \alpha)^{1/2}} \] (56)

An expression for the intermediate range of bubble diameters is given in [15]. The reason for the different expressions for the drift velocity in the "laminar" bubble regime stems from the fact that the drag on a single bubble depends on the bubble size. A more detailed discussion and a comparison with experimental data are given in [15].

It can be seen from the preceding that, in the bubbly and the slug flow regimes, the local drift velocity can be expressed in a form of

\[ V_{lg} = v_r (1 - \alpha)^k \] (57)

where \(v_r\) is the terminal rise velocity of a single bubble in an infinite medium, and the value of the exponent changes from \(k = 0\) to \(k = 3\), depending on the bubble size.

In order to evaluate now the effect of the drift velocity on the average volumetric concentration, we consider the second term on the right-hand side of equation (57); thus from (34) and (57)

\[ \frac{\langle \alpha V_{lg} \rangle}{\langle \alpha \rangle} = \frac{1}{\langle \alpha \rangle A} \int_A v_r (1 - \alpha)^k dA \] (58)

For the slug flow and for the turbulent bubbly flow, we obtain particularly simple expressions for the weighted mean drift velocity; thus from equations (57), (54), and (58), we obtain

\[ \frac{\langle \alpha V_{lg} \rangle}{\langle \alpha \rangle} = 0.35 \left[ \frac{g \Delta \rho D}{\rho_l} \right]^{1/4} \] (59)

for the slug flow regime and

\[ \frac{\langle \alpha V_{lg} \rangle}{\langle \alpha \rangle} = 1.53 \left[ \frac{g \Delta \rho}{\rho_l^2} \right]^{1/4} \] (60)

for the bubbly churn-turbulent regime.

In order to evaluate the weighted mean drift velocity for the cases described by equations (55) and (56), we assume again a volumetric concentration profile given by equation (44). The weighted mean drift velocity then follows from equations (31) and (44); thus

\[ \frac{\langle \alpha V_{lg} \rangle}{\langle \alpha \rangle} = \frac{1}{\langle \alpha \rangle A} \int_A (1 - \alpha)^k dA \] (61)

where

\[ \langle \alpha \rangle = \frac{1}{\pi R^2} \int_0^R [\alpha_r - (\alpha_i - \alpha_r) e^{-x^2/2}] 2\pi r dr = \frac{\alpha_r n}{n + 2} + 2\alpha_u \] (62)

The integral on the right-hand side of equation (61) can be put in the form of

\[ I_K = \int_0^1 (\alpha + bx^n)^{1/n} dx \] (63)

whose solution is

\[ I_K = \left( \frac{a + b}{nk + 2} \right)^{1/n} + \frac{nk}{nk + 2} I_{K-1} \] (64)

For the special case when \(\alpha_u = 0\), the results are shown in Fig. 2. It can be seen that, when the drift velocity depends on the concentration, the value of the weighted mean drift velocity is smaller than that corresponding to the slug flow or to the turbulent bubbly flow.

Discussion of Analytical Results

Summary of Analytical Results

Before proceeding with a discussion, it will be helpful to summarize the main results of this analysis.

The average velocity of phase two is given by equation (32), thus

\[ \langle v_2 \rangle = \langle j \rangle + \langle V_{lg} \rangle \] (36)

whereas the weighted mean velocity is given by equation (37), thus

\[ \langle v_2 \rangle / \langle \alpha \rangle = \langle \alpha v_2 \rangle / \langle \alpha \rangle = C_v \langle j \rangle + \langle \alpha V_{lg} \rangle / \langle \alpha \rangle \] (32)

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In equation (37), the distribution parameter $C_0$ takes into account the effect of nonuniform flow and concentration profiles, whereas the weighted mean drift velocity $\langle \alpha V_{st} \rangle / \langle \alpha \rangle$ accounts for the effect of the local relative velocity between the phases.

The dimensionless form of equation (37) is given by equation (39), i.e., by

$$\frac{\langle \beta \rangle}{\langle \alpha \rangle} = C_0 + \frac{\langle \alpha V_{st} \rangle}{\langle \alpha \rangle \langle j \rangle}$$

whence one obtains equation (41), which is the general expression for the average volumetric concentration $\langle \alpha \rangle$ and is valid for any two-phase flow regime; thus

$$\langle \alpha \rangle = \frac{\langle \beta \rangle}{\langle \alpha \rangle}$$

The velocity ratio given by equation (42) then follows from equations (39) and (41); thus

$$\frac{\bar{\nu}_2}{\bar{\nu}_1} = \frac{1 - \alpha}{\alpha}$$

For axisymmetric vertical flow, the expressions for the distribution parameter $C_0$ are given by equations (46) and (48). In general, the drift velocity $V_{st}$ is a function of concentration in which case the weighted mean drift velocity is given by equations (61) and (62). Especially simple expressions result when the drift velocity is constant, as in the case of churn-turbulent bubbly flow, of slug flow, and of a pseudo jet flow (which is discussed in the section on "Steam-Water Systems in Vertical Flow With a Free Interface at High Pressure" of this paper).

The weighted mean velocity of the gas for the churn-turbulent bubbly regime follows then from equations (37) and (60); thus

$$\bar{\nu}_2 = \frac{\langle \beta \rangle}{\langle \alpha \rangle} = C_0 j_2 + 1.53 \left( \frac{\rho g \Delta \rho}{\rho_1} \right)^{1/4}$$

where, for the numerical constant, we have used the value of 1.18 recommended by Harmathy [22]. We could have used instead the value 1.18 proposed originally by Peebles and Garber. For the slug flow regime, we obtain from equations (37) and (59) the following expression:

$$\bar{\nu}_2 = \frac{\langle \beta \rangle}{\langle \alpha \rangle} = C_0 j_2 + 0.65 \left( \frac{\rho g \Delta \rho}{\rho_1} \right)^{1/2}$$

The weighted mean velocity of the gas in the pseudo jet flow regime is derived in the section on "Steam-Water Systems."

**Method of Analysis: The $\bar{\nu}_2 - (j)$ Plane**

It was noted that the results of this analysis are general and can be applied to any two-phase flow regime. Perhaps the greatest value of the result lies in the method of analyzing and correlating experimental data implied by equation (37). It can be seen that this equation suggests a plot of the weighted mean velocity $\langle j_2 \rangle / \langle \alpha \rangle$ versus the average volumetric flux density $j$ of the mixture. Such a mean velocity-flux density plane is shown in Fig. 3; it has the following important characteristics:

(a) If the concentration is uniform across the duct, then the value of the distribution parameter $C_0$ in equation (37) is equal to unity (see equations (47) and (48)). If, in addition, one neglects the effect of the local relative velocity $v_r$, i.e., if one sets the value of the local drift velocity $V_{st}$ equal to zero, then equation (37) plots into a straight line through the origin having an angle of 45 deg. This line represents the locus of points of

$$\frac{\langle j_2 \rangle}{\langle \alpha \rangle} = \frac{\langle j \rangle}{1 - \alpha}$$

In the literature, such a flow is referred to as the homogeneous flow.

(b) If the concentration across the duct area is not uniform, then the value of the distribution parameter $C_0$ is larger than unity for axisymmetrical distributions with the highest concentration at the axis of symmetry. (See equations (45), (49), and (50).) If the concentration is the highest near the duct wall (as in subcooled boiling), the value of $C_0$ is less than unity (see equation (52)). Consequently, the slopes of the lines in the velocity-flux plane reflect the effect of the nonuniform flow and concentration profiles.

(c) When the distribution parameter $C_0$ is larger than unity and the weighted mean drift velocity is not neglected, then the value of the weighted mean velocity $\langle j_2 \rangle / \langle \alpha \rangle$ will be larger than $\langle j \rangle$ (see equation (37)). In this case, the data will plot above the homogeneous flow line. Furthermore, it can be seen from equation (42) that the slip velocity ratio also will be larger than unity. Consequently, the weighted mean velocity of the gas is larger than that of the liquid, i.e.,

$$\frac{\bar{\nu}_2}{\bar{\nu}_1} > \frac{\langle \alpha \rangle}{\langle j \rangle}$$

The opposite is true for values of $\langle j_2 \rangle / \langle \alpha \rangle$ which plot below the homogeneous flow line.
Particularly simple results are obtained for two-phase flow regimes with fully established flow and concentration profiles when the local drift velocity is constant. For such flow regimes, equation (37) plots into a straight line.10 For a particular two-phase flow pattern, the slope of such a straight line gives the value of the distribution parameter $C_0$, whereas the intercept of this line with the $\langle j \rangle / \langle \alpha \rangle$ axis gives the value of the weighted mean drift velocity $\langle \alpha V_{z\alpha} \rangle / \langle \alpha \rangle$.

(c) If the weighted mean drift velocity $\langle \alpha V_{z\alpha} \rangle / \langle \alpha \rangle$ is much smaller than the average volumetric flux density for the mixture $\langle j \rangle$, i.e., when

$$\frac{\langle \alpha V_{z\alpha} \rangle}{\langle \alpha \rangle} < 1$$

then the effect of the local relative velocity on the average volumetric concentration $\langle \alpha \rangle$ can be neglected when compared to the effects of nonuniform flow and concentration distributions (see equation (41)).

(f) It can be expected that changes of flow regimes will result in a change of the value of the distribution parameter $C_0$. This statement follows from the fact that a change of flow regime implies a change of the flow and concentration profiles as well as of the geometry of the interface between the two phases.

(g) Since the drift velocity depends on the momentum transfer between the two phases, it depends, therefore, on the stress fields in both phases and on the geometry of the interface. Consequently, it can be expected that the value of the drift velocity will change as soon as the flow regime has changed.

(h) In view of items (f) and (g), it can be expected that plots in the mean velocity-flux density plane which show abrupt changes of slope and of intercept may be interpreted as indicating a change of flow regimes.

(i) For two-phase systems with heat and/or mass addition along the duct, the two-phase flow regime will change resulting in a corresponding change of $C_0$ along the duct. Consequently, for such systems, plots in the $\langle j \rangle$/$\langle \alpha \rangle$ plane will exhibit a curvature. For axisymmetric vertical flows with high concentration along the axis, the variation of curvature will not be very pronounced because, for such flows, the value of $C_0$ can vary from $C_0 = 1.0$ to $C_0 = 1.5$ (see Fig. 1).

The $\langle \alpha \rangle$ — $\langle \beta \rangle$ Plane

It was seen that equation (37) suggested a $\hat{n} - \langle j \rangle$ plane for analyzing experimental data. We note now that equation (39) suggests another plane, i.e., the $\langle \alpha \rangle$ — $\langle \beta \rangle$ plane which is plotted in Fig. 4. The various regions that are indicated are self-explanatory and can be easily deduced from equations (39) and (42). This $\langle \alpha \rangle$ — $\langle \beta \rangle$ plane was used rather extensively by Russian investigators6 to correlate and analyze their data; it is discussed also in [4 and 5].

By examining and comparing the $\hat{n} - \langle j \rangle$ and the $\langle \alpha \rangle$ — $\langle \beta \rangle$ planes, we note that both planes show two regions that are separated by the homogeneous flow line. Thus both planes can be used to evaluate whether or not the slip velocity ratio is larger than unity. However, of the two planes, the $\hat{n} - \langle j \rangle$ is clearly superior because it correlates the data at low flow rates. This is not the case for the $\langle \alpha \rangle$ — $\langle \beta \rangle$ plane. It can be seen from equation (39) that, unless equation (60) is satisfied, the data, when plotted in the $\langle \alpha \rangle$ — $\langle \beta \rangle$ plane, will be represented by a family of curves with the flow rate as parameter.

It was noted that, at a given system pressure, equation (4) correlated experimental data with a single value of the constant $K$ which was independent of the flow rate. Thus, in the $\langle \alpha \rangle$ — $\langle \beta \rangle$ plane and in the region $0 < \beta < 0.5$, a single straight was a satisfactory approximation to the experimental data. This

10 Numerous statements, made in the literature, explain this linear relation as a special characteristic of the slug flow regime. It can be seen that this is not the case; it holds for other flow regimes as well.

11 We note, however, that the authors of [24] plotted their data on the $\hat{n} - \langle j \rangle$ plane.

Comparison With Previous Analyses

We shall compare now the results of this analysis with those discussed in the section, “Previous Work.”

(a) If we neglect the effect of the nonuniform flow and concentration profiles, then the distribution parameter is unity (see equation (45)) and equations (32) and (33) reduce to equation (1) proposed previously by Behringer [3]. In particular, with $C_0 = 1$, the equations for the churn-turbulent bubbly and for the slug flow regime, i.e., equations (65) and (66), become identical with equation (1). Note that, by definition, the average volumetric flux density for the mixture is given by

$$\langle j \rangle = \frac{Q_2}{A}$$

(b) If we neglect the effect of the local relative velocity between the phases, then it follows from equation (26), that the drift velocity $V_{z\alpha}$ and, therefore, the weighted mean drift velocity $\langle \alpha V_{z\alpha} \rangle / \langle \alpha \rangle$, is zero. It can be seen then that the slip velocity ratio given by equation (42) reduces to equation (3) proposed previously by Bankoff [4]. We note that the flow parameter $K$ in equation (3) is the inverse of the distribution parameter $C_0$ in equation (42).

(c) By comparing equation (4) with equation (39), it can be seen that if $\langle \alpha \rangle$ the drift velocity is zero as implied by Bankoff's analysis or (b) if the ratio of the weighted mean drift velocity to the average volumetric flux density of the mixture $\langle j \rangle$ is small, i.e., if equation (60) is satisfied, then equation (39) reduces to equation (4). It can be also seen that if equation (60) is not satisfied, then an equation of the type of equation (4) will not be in agreement with experimental data; i.e., the ratio $\langle \alpha \rangle / \langle \beta \rangle$ will not be independent of the mass flow rate. This is indeed the case at low volumetric flow rates as observed and discussed previously.
by several Russian investigators (see for example [24]) as well as by Neal [11].

\( \text{(d) If we neglect the effect of the nonuniform flow and concentration profiles, i.e., if we set } C_0 = 1, \text{ then equation (66) reduces to the slug flow equation proposed previously by Griffith and Wallis [7] (see discussion in conjunction with equation (6)).} \)

\( \text{(e) By comparing equation (7) with equation (66), we see that both are of the same form. We note, however, that equation (7) was not derived either in [10] or in [11]. Whereas the authors of [10] used plausible arguments to modify the analysis of Griffith and Wallis [7], the author of [11] adds without proof, i.e., adds arbitrarily, the second term on the right-hand side of equation (7) in order to modify the analysis of Bankoff [4]. Furthermore, whereas Niklin, Wilkes, and Davidson [10] consider the velocity } v_2 \text{ in equation (7) to be the actual velocity of vapor slug, Neal [11] considers it to be "the total cross-sectional average gas velocity."} \)

We note that the average velocity obtained by integration over the cross-sectional area of the duct (see equation (32)) is not equal to the weighted mean velocity given by equation (38). This statement becomes obvious if one recalls that the average of a ratio is not equal to the ratio of the averages, i.e.,

\[
\langle \frac{v_1}{v_2} \rangle = \left( \frac{\langle j_x \rangle}{\langle j_y \rangle} \right) \neq \left( \frac{\langle j_x \rangle}{\langle j_y \rangle} \right) = \langle \alpha \rangle \tag{71} \]

The coefficient \( C_0 \) in equation (7) can be different from unity only if one formulates the problem in terms of the weighted mean velocity given by equation (37).

We note, further, that neither the analysis of [10] nor that of [11] can predict the decrease of the slip velocity ratio below unity, which is one of the results derived in this paper (cf. items (b) and (c) in the section, "Method of Analysis").

**Comparison With Experiments**

**Effect of Concentration Profiles**

The comparisons of the results predicted by the analysis with experimental data are made for fully developed, adiabatic, vertical flow through a round duct. These data fall generally in the bubbly, slug, and two-phase jet flow regimes.

It should be noted here that a great deal of data was encountered in the literature that could not be used to check against this analysis because the experiments were either improperly run or the data improperly recorded (or both).

In running experiments of this type, two considerations are of great importance. They are: (a) The method by which the gas or vapor is introduced into the pipe, i.e., size of orifices, etc., and (b) whether or not the flow regime is fully developed or is in the process of developing. These two items can cause a good deal of trouble in interpreting the experimental results and may lead to misconceptions.

A further difficulty arises when the experimental results are analyzed for determining the effects of the concentration profile and of the flow profile of the mixture on the value of \( \alpha \). The authors were not able to find a single reference reporting experiments where these two profiles have been recorded simultaneously. Data are, however, available on concentration profiles only; consequently, we can examine the validity of the assumed distribution given by equation (44).

Fig. 5 shows the experimental data of Petrick [25] for an adiabatic steam-water mixture at 600 psi flowing through a pipe \((D = 2 \text{ in.) together with the results predicted by equation (44). The values of the exponent } n \text{ which were used in equation (44) were determined from equation (62) by using the measured value of } \alpha_0 \text{ and } \langle \alpha \rangle ; \text{ the value of } \alpha_0 \text{ was zero since the flow was adiabatic. It can be seen from Fig. 5 that the assumed distribution is in agreement with the data. The reader will note also that, as the volumetric flux density of the mixture } (i.e., j) \text{ increases, the value of the exponent } n \text{ increases, resulting in flatter profiles. Similar experimental results, which justify the use of equation (62) for approximating the concentration profiles, are given in [26].} \)

It was noted previously that, apparently, experiments have not been conducted yet in which the concentration and the mixture flow profiles were measured simultaneously. Consequently, no data are available for determining the value of the distribution parameter \( C_0 \) (given by equation (45)) from the measurements of the two distributions. An estimate of the value of \( C_0 \) can be made, however, if we assume that the flow profile of the mixture is similar to that of the volumetric concentration, i.e., that the two exponents \( m \) and \( n \) in equations (43) and (44) are equal. This assumption does not appear unreasonable if one considers that the volumetric flux density of the mixture \( j \) will be greatly (if not
mostly) affected by the volumetric flow of the gas. With this assumption, and for adiabatic flow \((\alpha_a = 0)\), equation (45) reduces to

\[
C_v = \frac{n + 2}{n + 1}
\]

(72)

Fig. 8 shows the value of \(C_v\) computed from equation (72) for the experimental data shown in Fig. 5. With the exception of the run at the lowest volumetric concentration, i.e., \((\alpha) = 0.189\), it can be seen that, as the profiles become flatter (both \(n\) and \(j\) increase), the value of \(C_v\) decreases, which is in agreement with the analytical results plotted in Fig. 1. The average value of \(C_v\) for these experiments is approximately \(C_v = 1.35\). Inserting this value of \(C_v\) into equation (65) gives the weighted mean gas velocity that is plotted and compared to the experimental data in Fig. 6. The agreement appears reasonable. The reader will note that the effects of the distribution parameter and of the gas weighted mean drift velocity as well as the characteristics of the velocity-flux phase (discussed in the preceding section) are supported by the results shown in this figure. It is apparent also that, for a definitive confirmation of these results, experimental data are needed on simultaneously recorded concentration and flow profiles.

Two-Component Systems

In Fig. 7 are plotted the experimental results for an air-water mixture flowing through a circular pipe (5.5 in. ID) reported by Petrick [23] (approximately 120 data points are plotted). These experiments are of special value because the volumetric concentration \((\alpha)\) was determined together with the bubble size distribution as a function of flow rates. Thus, these experiments provide quantitative data on the average volumetric concentration, bubble size distribution, and on the flow regime (only the bubbly flow regime was observed). It can be seen from Fig. 7 that the data, when plotted in terms of the coordinate system suggested by equation (27), show indeed a linear relation with respect to the average volumetric flux density \((j)\) of the mixture. The slope of the line is equal to \(C_v = 1.6\), indicating a higher concentration of bubbles in the central region of the pipe (cf. the results plotted on Fig. 2). Also, the value of the intercept of this straight line with the \((j)/\alpha\) axis is in agreement with the value of the weighted mean drift velocity for the churn-turbulent bubbly flow predicted by equation (60). It appears, therefore, that the results predicted by equation (65) are in agreement with these experiments.

We note that, for a pipe diameter equal to \(D = 5.5\) in. = 14 cm, the drift velocity for slug flow, predicted by equation (59), is equal to approximately 45 cm/sec. Consequently, if the two-phase mixture were in the slug flow regime, the intercept should have had a value of 45 cm/sec instead of 25 cm/sec as predicted by equation (60). Since no slugs were observed in Petrick’s experiments but only bubbles, the intercept is in agreement with the predicted drift velocity for the churn-turbulent bubbly flow, i.e., with equation (60). This fact confirms our previous statement that a linear relation between \((j)/\alpha\) and \((j)\) does not imply that the mixture is in the slug flow regime; statements to this effect, which have been often made in the literature, are obviously incorrect. This distinction between the slug flow and the churn-turbulent flow is discussed in more detail in [15], together with the other regimes of bubbling.

The apparent reason for identifying the churn-turbulent bubbly regime with the slug flow regime arises probably from the fact that, for water at atmospheric pressure flowing through pipes with diameter in the range from 1 to 2 in., the drift velocity for slug flow predicted by equation (59) has practically the same value as that predicted by equation (60) for the bubbly flow. Thus, unless visual observations are made, a differentiation between the two flow regimes cannot be made. This statement is illustrated in Fig. 8, which shows the experimental data of Smissonert [27] for air-water flowing through a 2 in. ID pipe. For this pipe size, the drift velocity for slug flow, predicted by equation (59), is equal to approximately 20 cm/sec, whereas for the bubbly flow, predicted by equation (60), it is approximately equal to 25 cm/sec. Consequently, the difference is too small for an accurate differentiation between the two flow regimes. This can be seen in Fig. 8, which shows data for air-water mixture for various air and liquid flow rates. Here, again, the linear relation is striking; the slope is equal to \(C_v = 1.2\), indicating a flat profile (see Fig. 2). The intercept is in the range of 25-30 cm/sec in agreement with either equation (59) or equation (60). The data of Smissonert obtained with nitrogen-mercury mixtures are not plotted because of the large scatter. It is possible that in these experiments an axisymmetric flow pattern of the mixture was not established or that the experiments were not sufficiently controlled.

In order to investigate further the difference between the slug flow and the bubbly flow regime, we consider the experimental results reported by Bailey, Zmola, and coworkers [29] and plotted in Figs. 9 and 10. These investigators observed two flow regimes; for concentration smaller than 0.13, the flow was in the bubbly regime; whereas for concentration in excess of \((\alpha) = 0.286\), the flow was in the slug regime; a transition region existed between \((\alpha) = 0.15\) and \((\alpha) = 0.28\). They noted also that, in the bubbly regime, the diameter of the pipe did not have an effect on the value of \(\alpha\) or on the bubble rise velocity, whereas in the slug flow regime, both \((\alpha)\) and \(v_2\) were affected by the diameter. It can be seen from Fig. 9 that the values of \((j)\) larger than approximately 20 cm/sec, i.e., for \((\alpha)\) larger than 0.27, the data are

\[\text{Fig. 9 Comparison of values predicted by equation (66) with experimental data of Bailey, et al. [29]}\]
Fig. 10 Comparison of values predicted by equation (65), i.e., with experimental data of Bailey, et al. [29], for air-water mixtures. This figure also shows transition from "churn-turbulent" bubbly flow to slug flow regime.

Fig. 11 Plot of data of [30] for different flow regimes

Air-water mixtures

- $D = 6" \times 15.3$ cm
- $D = 12" \times 30.4$ cm
- $D = 24" \times 61.0$ cm

It is seen that, when the average volumetric concentration ($\alpha$) is smaller than approximately 0.26, all experimental points, obtained with various pipes, can be predicted by equation (65), which for a batch process can be written also as

$$\langle j_\alpha \rangle = 1.53 \left[ \frac{\sigma g \Delta \rho}{\rho_i^2} \right]^{1/3} \frac{\langle \alpha \rangle}{1 - C_0(\alpha)}$$  \hspace{1cm} (73)$$

It is of the form discussed previously by Zuber and Hench [15]. It can be seen further that in this region there is no effect of pipe diameter, which is in agreement with the experimental observation of Bailey, Zmola, and coworkers. On the same figure, we have plotted also the values predicted by the equation for slug flow, in a batch process, i.e., by:

$$\langle j_\alpha \rangle = 0.35 \left[ \frac{\sigma g \Delta \rho D}{\rho_i} \right]^{1/3} \frac{\langle \alpha \rangle}{1 - C_0(\alpha)}$$  \hspace{1cm} (74)$$

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It can be seen that, in the churn-turbulent bubbly regime, and for a given value of \( \beta \), the equation for slug flow underestimates, considerably, the value of \( \alpha \). However, as \( \beta \) increases and \( \alpha \) increases beyond a value of approximately 0.26, the data start deviating from the line for the bubbly flow regime and, after a transition region, they merge with the curves for the slug flow regime (cf. Fig. 9).

In Fig. 11 are plotted the experimental data of Wallis, et al. [30], for air-water mixture at atmospheric pressure flowing through a 0.975 in. ID pipe. It can be seen that the data can be approximated, rather closely, by two straight lines—one corresponding to the slug flow regime and the other to the annular flow regime. The straight line for the slug flow regime is that predicted by equation (66). An analysis of the annular flow regime will lie presented in a forthcoming paper. Here we note that, for an established annular flow, the data plot is a straight line with a value of the distribution parameter equal to unity. The reader should note also the change of the values of the distribution parameter \( C_0 \) and of the weighted mean drift velocity with a change of flow regime.

It appears from the foregoing that the conclusions which have been enumerated in “Method of Analysis” under items (a) through (i) are supported by experimental data. These and similar results (soon to be reported) confirm the generality of the analytical results and of the method of analysis proposed in this paper.

**Steam-Water Systems in Vertical Flow With a Free Interface at High Pressure**

Another problem of practical importance is concerned with the relation between the volumetric vapor concentration \( \alpha \) and the volumetric vapor flux density \( \beta \) for a steam-water mixture with a free interface in a vertical container of large diameter. Such a system may approximate the region above a reactor core. Results of several experiments concerned with this problem have been reported in the literature [24, 31, 32] and are reproduced in Figs. 12, 13, and 14.

It can be seen from Figs. 12–14, that, after a short transition region (for low value of \( \beta \)) and within the accuracy of these experiments performed at high pressure, the data follow the linear relation given by equation (37). The slopes of these lines vary between approximately \( C_0 = 1.0 \) and \( C_0 = 1.2 \) for almost all runs. However, at comparable pressures, the values of the weighted mean drift velocities as determined from the intercepts of these straight lines change from one group of experiments to another. Thus it appears that, with different apparatus, different values are obtained for the weighted mean drift velocity.

In order to clarify this new aspect of the problem, we note first that neither the weighted average drift velocity given by equation (59) nor that given by equation (60) is in agreement with the values of the intercepts shown in Figs. 12–14. The values predicted by these two equations do not show such a strong dependence upon pressure as indicated by the intercepts shown in Figs. 12–14. Consequently, in view of the discussion given in the preceding section, it would appear that the flow was neither in the churn-turbulent bubbly nor in the slug flow regime.

We are led then to consider the influence of the mode of gas injection on the volumetric concentration in a batch bubbling system. It becomes evident that, for the same volumetric flow rate of the gas \( Q_2 \) (not excessively high), different results will be obtained depending on whether the gas is injected through one large orifice or through a porous plate. This statement follows from the fact that different regimes exist during the process of bubbling from an orifice. At low flow rates, single bubbles are generated whereas, at high flow rates, large bubbles and gas jets...
are formed [33]. Since the mode of gas injection is the boundary condition imposed on the bubbling field, it can be expected that it plays an important role. However, very few experimenters describe or indeed pay any attention to this aspect of the problem. Consequently, contradictory results and incorrect conclusions are often being made in the literature. The importance of the mode of gas injection in a bubbling batch system is discussed further in [28].

In connection with the experiments of [24, 31, and 32] we note that (a) the steam was introduced into the water through a perforated plate and (b) the liquid level above the plate did not exceed 3 ft, i.e., 1 meter. It is possible that, under these conditions, the flow field was strongly influenced by the boundary conditions imposed by the steam jets issuing from the perforations on the plate. Such a flow regime would consist of a steam-water jet rising in the core above the perforation with a downflow of liquid along the walls of the container. This flow regime would be greatly influenced by the initial height of the liquid, by the number and size of perforations, as well as by the dimensions of the container. With the exception of the dimensions of the container, no information is available in the literature relative to the other parameters. Consequently, a detailed quantitative investigation is not warranted at this time.

However, we can explore the problem a little further by noting that, if the flow is influenced by the vapor jets issuing from the perforations in the plate, then the kinetic energy of the vapor, the surface tension, and the geometry of the perforations will influence the process. It appears, therefore, reasonable to formulate the problem in terms of a critical Weber number based on the orifice diameter and express it as a function of a geometry group represented by the Eötvös number; thus

\[ \frac{\rho g \sigma^2 d_0}{\sigma} = \frac{a}{\bar{V}^2} \left( \frac{\rho g \Delta \rho \bar{V}}{\sigma} \right)^{1/4} \]

where \( \bar{V} \) is the velocity of the gas through the orifice of diameter \( d_0 \). In absence of data on liquid height and on the number and size of perforations, the simplest relation between the weighted mean drift velocity and equation (75) is of the form

\[ \frac{\bar{V}}{\bar{\alpha}} = A_p \]

where, if the preceding arguments are correct, the parameter \( A_p \) should not depend on pressure but may depend on the characteristic, i.e., geometry, of the apparatus. We can then test the preceding results by investigating whether or not the values of the intercepts determined from Figs. 12-14, when inserted in equation (76), give, for a given apparatus, a value of the constant \( A_p \) which is approximately independent of pressure.

In Tables 1 through 3, we show the values of the weighted average drift velocities determined from the intercepts of the straight lines shown in Figs. 12-14; we also show the values of the constant \( A_p \) determined from equation (76) for the indicated system pressures.

It appears from Tables 1-3 that, for a given apparatus, and as a first approximation at least, the value of the parameter \( A_p \) in equation (76) remains constant and independent of pressure. It also can be seen from these tables that this value changes with a change of system, i.e., of apparatus. Such a change can be explained in terms of changes of container geometry, liquid height, and of the number, distribution, and sizes of the orifices in the perforated plate. Additional comparisons are shown in [28].

### Table 1

<table>
<thead>
<tr>
<th>( P ), psia</th>
<th>( P ), atm</th>
<th>( \bar{V}/\bar{\alpha} ), cm/sec</th>
<th>( A_p ), equation (76)</th>
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<td>600</td>
<td>408</td>
<td>0.65</td>
<td>0.79</td>
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<tr>
<td>800</td>
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<td>0.81</td>
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<tr>
<td>1000</td>
<td>68</td>
<td>0.45</td>
<td>0.82</td>
</tr>
<tr>
<td>1200</td>
<td>82</td>
<td>0.40</td>
<td>0.845</td>
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<tr>
<td>1400</td>
<td>95</td>
<td>0.34</td>
<td>0.840</td>
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<tr>
<td>2000</td>
<td>136</td>
<td>0.25</td>
<td>0.89</td>
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</table>

### Table 2

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<tr>
<th>( P ), psia</th>
<th>( P ), atm</th>
<th>( \bar{V}/\bar{\alpha} ), cm/sec</th>
<th>( A_p ), equation (76)</th>
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<td>88</td>
<td>6</td>
<td>0.38</td>
<td>0.530</td>
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<td>250</td>
<td>17</td>
<td>0.35</td>
<td>0.500</td>
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<tr>
<td>485</td>
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<td>0.338</td>
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<tr>
<td>880</td>
<td>60</td>
<td>0.20</td>
<td>0.610</td>
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<tr>
<td>1140</td>
<td>77</td>
<td>0.16</td>
<td>0.640</td>
</tr>
<tr>
<td>1350</td>
<td>92</td>
<td>0.12</td>
<td>0.650</td>
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### Table 3

<table>
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<th>( P ), psi</th>
<th>( \bar{V}/\bar{\alpha} ), cm/sec</th>
<th>( A_p ), equation (76)</th>
</tr>
</thead>
<tbody>
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<td>250</td>
<td>0.421</td>
<td>0.608</td>
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<tr>
<td>530</td>
<td>0.508</td>
<td>0.650</td>
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<td>1040</td>
<td>0.619</td>
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<td>1620</td>
<td>0.640</td>
<td>0.635</td>
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<td>2060</td>
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<td></td>
</tr>
<tr>
<td>2650</td>
<td>0.650</td>
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</table>

Fig. 14 Comparison of values predicted by equation (77) with experimental data of [24] for steam-water mixture at various pressures. 

Transactions of the ASME
In view of the foregoing, it would appear that equation (37), with the values of the weighted average drift velocity determined from equation (76), i.e., that the following equation
\[
\left( \frac{\langle j \rangle}{\langle \alpha \rangle} \right) = C_0(j) + A_p \left( \frac{\sigma_j \delta \rho_j}{\rho_\alpha^2} \right)^{1/3}
\]  
(77)
can be used to scale the effect of pressure for a given apparatus. It is evident also that additional experimental data are needed in order to clarify several unresolved questions relative to the effects of container geometry, of initial liquid height, of the number of perforations in the plate, as well as of their size distribution and of their spacing.

Steam Water Systems at High Pressures in Forced Convection Through Circular Ducts

The results of an extensive study of the effect of the Froude number on the relation between \(\langle j \rangle\) and \(\langle \alpha \rangle\) are reported in [34, 35]. The Froude number was defined there as
\[
N_{Fr} = \frac{\rho_j \langle j \rangle^2}{g \delta \rho_j D^2}
\]  
(78)
It was varied from 7.5 to 2270. The experiments were conducted with two pipe sizes, \(D = 1.7\) and \(D = 3.0\), at four pressure levels, \(P = 20, 40, 270,\) and 120 atm.

In order to compare the results predicted by the analysis of this paper with these experimental results, we use the definition of the Froude number given by equation (78) and rewrite equations (66), (65), and (77) in the following forms, respectively:
\[
\left( \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right) = C_0 + 0.35 \left( \frac{1}{N_{Fr}} \right)^{1/2}
\]  
(79)
\[
\left( \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right) = C_0 + 1.53 \left( \frac{1}{N_{Fr}} \right)^{1/2} \left( \frac{\sigma}{g \delta \rho_j D^2} \right)^{1/4}
\]  
(80)
and
\[
\left( \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right) = C_0 + A_p \left( \frac{1}{N_{Fr}} \right)^{1/2} \left( \frac{\sigma}{g \delta \rho_j D^2} \right)^{1/4} \left( \frac{\rho_j}{\rho_\alpha} \right)^{1/2}
\]  
(81)
It can be seen from the preceding equations that, as the Froude number increases, the ratio \(\langle \beta \rangle/\langle \alpha \rangle\) tends to the value of \(C_0\). We have shown already that, for axisymmetric vertical flow when \(\alpha_w > \alpha_o\), the distribution parameter \(C_0\) varies from approximately \(C_0 = 1.5\) to \(C_0 = 1.0\).

The results of [34, 35] show that, up to approximately \(\langle \beta \rangle = 0.7\), the value of the ratio \(\langle \beta \rangle/\langle \alpha \rangle\) remains constant in agreement with the previous results of Armand. However, in contrast to the results of Armand [6], the value of this ratio is function of both pressure and of the Froude number. The values of the ratio \(\langle \beta \rangle/\langle \alpha \rangle\) determined from these experiments as function of the Froude numbers and of the system pressure are tabulated in Table 4.

It can be seen from Table 4 that, as the pressure and/or the Froude number increase, the value of the ratio \(\langle \beta \rangle/\langle \alpha \rangle\) tends to unity. This is in agreement with our discussion of the distribution parameter \(C_0\), which tends to unity for flat turbulent profiles which can be expected at high Froude numbers.

In order to determine the variation of the flow parameter \(C_0\) with pressure and flow, it is necessary to consider the momentum equation because the flow and concentration profiles depend on the dynamic conditions in the mixture. The problem can be solved either by modifying the approach presented by Levy [36] or by using a different formulation; this aspect of the problem will be treated in a future publication. Here we note that data on the flow and concentration distributions as function of the volumetric flow rate, pressure, properties, and geometry are almost nonexistent. It is evident that an accurate prediction of \(\langle \alpha \rangle\), as well as a sound analysis concerned with predicting the flow and concentration profiles, will depend on the availability of such data.

Clearly, the value of \(C_0 = 1.2\) is only a rough approximation as indicated by the values tabulated in Table 4.

Conclusions

1 A general expression (equations (37) or (39) or (41)) applicable to any two-phase flow regime which can be used either for predicting the average volumetric concentration \(\langle \alpha \rangle\) or for analyzing and interpreting experimental data is derived.

2 The analysis takes into account both the effect of nonuniform flow and concentration profiles as well as the effect of the local relative velocity between the phases.

3 The effect of the nonuniform flow and concentration profiles is taken into account by the distribution parameter \(C_0\) in equation (37). For fully established profiles, the value of \(C_0\) remains constant.

4 The effect of the local relative velocity and the concentration profile is taken into account by the weighted mean drift velocity \((\alpha V_\alpha)/\langle \alpha \rangle\) in equation (37).

5 For the axisymmetric vertical flows through circular ducts which were considered in this paper, it is shown that the value of the distribution parameter depends on the exponents of the flow and concentration profiles as well as on the value of the volumetric concentration evaluated at the wall \(\alpha_w\) and at the centerline \(\alpha_c\) of the duct (cf. equations (45) and (56)).

6 The value of the distribution parameter can be less than one, i.e., \(C_0 < 1\) when \(\alpha_w > \alpha_o\) or larger than one, i.e., \(C_0 > 1\) when \(\alpha_w < \alpha_o\). In the latter case, it is shown that the value of the distribution parameter varies between \(C_0 = 1.0\), for flat profiles, and \(C_0 = 1.5\), for peaked profiles (cf. Fig. 1).

7 It is shown that, at high mass flow rates, the velocity ratio is larger than one, i.e., \(\beta_\alpha/\beta_\alpha < 1\) when \(C_0 > 1\), or smaller than one, i.e., \(\beta_\alpha/\beta_\alpha > 1\) when \(C_0 < 1\).

8 Two types of flow regimes were considered on paper. For the first group, which includes \(\langle \alpha \rangle\) the “churn-turbulent” bubbly regime, \(\langle \beta \rangle\) the slug flow regime, and \(\langle \gamma \rangle\) the two-phase jettype flow regime, the local drift velocity, and, therefore, the weighted average drift velocity, does not depend upon volumetric concentration (cf. equations (50), (60), and (76)). For the second group, which includes the various types of the “ideal bubbly regime,” the weighted average drift velocity is a function of the average volumetric concentration (cf. equation (38)).

9 The results indicate (see equation (37)) that a very useful method for analyzing and correlating experimental results is by plotting the data in the weighted mean velocity-average volumetric flux density plane, i.e., in the \(\beta_\alpha - \langle \gamma \rangle\) plane. The important characteristics of this plane are discussed.

10 It is shown that, when the drift velocity does not depend upon concentration, particularly simple expressions are obtained for two-phase flow regimes with fully established flow concentration profiles. For such flows, the slopes of the straight lines in the \(\beta_\alpha - \langle \gamma \rangle\) plane give the values of the distribution parameter, whereas the intercepts with the \(\beta_\alpha\) axis give the value of the weighted mean drift velocity for the particular flow regime.

11 Plots in the velocity-flux plane which show abrupt change of slope and of intercept can be interpreted as indicating a change of flow regime.

12 Since in two-phase flow systems with heat and/or mass transfer
addition along the duct, the flow and concentration profiles constantly change, it can be expected that the curves which approximate experimental data in the velocity-flux plane will exhibit a slight curvature.

13 The comparison with experimental data, for various flow regimes, shows that the various results predicted by the analysis are in qualitative and quantitative agreement with experiments (cf. Figs. 5-14).

14 It is amazing that so many correlations have been obtained and so many analyses of this problem have been conducted by means of computers when data shown in Figs. 6 through 14 exhibit such a striking linear dependence.

Recommendations

The results of this analysis indicate that, for a detailed understanding of the phenomenon, simultaneously recorded data on both the flow and concentration profiles are required.

Acknowledgment

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17 N. Zuber and N. Fitoroy, "Steady-State, Transient Response and Operating Limits of Liquid-Liquid Fluidized Systems," to be published.